SOLVING HEAT CONDUCTION PROBLEMS BY THE STRAIGHT-LINES METHOD

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Boundary problems relating to the heat conduction equation are solved by the straight-lines method with spatial quantization.

The straight-lines method [2] was used in [1] for integrating equation

$$u'_t = u''_{xx} + f(x,\ t)$$

with various boundary and initial conditions, and with numerical results also shown.

In this article the straight-lines method will be applied to the solution of boundary problems relating to the heat conduction equation with variable coefficients which depend on the space variable.

Let the heat conduction equation be given for the interval $0 < x < b$

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) - q(x)u + f(x,\ t), \qquad \rho(x) > 0,\ k(x) > 0,\ q(x) \geqslant 0, \tag{1}$$

with the boundary conditions

$$u(0,\ t) = \varphi_1(t),\ u(b,\ t) = \varphi_2(t),\ 0 \leqslant t \leqslant T, \tag{2}$$

and with the initial condition

$$u(x,\ 0) = \psi(x),\ 0 \leqslant x \leqslant b. \tag{3}$$

When the interval $[0,\ b]$ is broken down into $n + 1$ arbitrary sections by straight-lines $x = x_i$ $(i = 1,\ 2,\ \ldots,\ n)$ in steps of $h_i = x_i - x_{i-1}$, and when the derivatives with respect to x are approximated by the difference expressions [3], we will obtain a system of ordinary differential equations

$$\rho_i\frac{du_i}{dt} = \frac{1}{\bar{h}_i}\left[\frac{k_{i+1/2}(u_{i+1} - u_i)}{h_{i+1}} - \frac{k_{i-1/2}(u_i - u_{i-1})}{h_i}\right] - q_i u_i + f_i(t) \tag{4}$$

$$(i = 1,\ 2,\ \ldots,\ n),$$

$$u_0 = \varphi_1(t),\ u_{n+1} = \varphi_2(t),\ 0 \leqslant t \leqslant T,$$

with the initial conditions

$$u_i(0) = \varphi_i\ (i = 1,\ 2,\ \ldots,\ n). \tag{5}$$

Here $h_i = 0.5(h_i + h_{i+1})$, $k_{i+1/2} = k(x_i + 0.5h_{i+1})$, $k_{i-1/2} = k(x_i = 0.5h_i)$, and $u_i = u_i(t)$ is the approximate solution of problem (1), (2), (3) on the straight-line $x = x_i$.

The problem (4), (5) will now be stated in vector form:

$$u' + Pu = f(t),\ u(0) = \psi, \tag{6}$$

where

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$$u = \begin{Vmatrix} u_1(t) \\ \cdot \\ \cdot \\ \cdot \\ u_i(t) \\ \cdot \\ \cdot \\ \cdot \\ u_n(t) \end{Vmatrix}, \quad f(t) = \begin{Vmatrix} f_1(t) - c_1\varphi_1(t) \\ f_2(t) \\ \cdot \\ \cdot \\ \cdot \\ f_i(t) \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1}(t) \\ f_n(t) - b_n\varphi_2(t) \end{Vmatrix}, \quad \psi = \begin{Vmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_i \\ \cdot \\ \cdot \\ \cdot \\ \psi_n \end{Vmatrix},$$

$$P = \begin{Vmatrix} a_1 & -b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -c_2 & a_2 & -b_2 & 0 & \dots & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \dots & -c_{n-1} & a_{n-1} & -b_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -c_n & a_n \end{Vmatrix},$$

$$b_i = k_{i+1/2}/\rho_i\hbar_i h_{i+1}, \quad c_i = k_{i-1/2}/\rho_i\hbar_i h_i, \quad a_i = b_i + c_i + q_i.$$

The matrix P appears to be a Jacobian [4], it has real and different eigenvalues, so that it may be written down as

$$P = G^{-1}\Lambda G,$$

where Λ is the diagonal n-th order matrix whose elements are the eigenvalues of matrix P.

If we designate

$$U = Gu, \quad F(t) = Gf(t), \quad \Psi = G\psi \tag{7}$$

and multiply on the left hand side of (6) by the matrix G, we will have

$$U' + \Lambda U = F(t), \quad U(0) = \Psi. \tag{8}$$

The spectrum of matrix P is determined from the characteristic polynomial, an expansion based on the following recurrence formulas:

$$\begin{gathered} D_0(\lambda) = 1, \quad D_1(\lambda) = a_1 - \lambda, \\ D_i(\lambda) = (a_i - \lambda)\ D_{i-1}(\lambda) - b_{i-1}c_i\ D_{i-2}(\lambda) \\ (i = 2,\ 3,\ \dots,\ n). \end{gathered} \tag{9}$$

The transformation matrix G^{-1} may embody the linearly independent eigenvectors (column vectors) of matrix P

$$X_s = (x_{1s},\ x_{2s},\ \dots,\ x_{is},\ \dots,\ x_{ns})' \quad (s = 1,\ 2,\ \dots,\ n),$$

which correspond to the various eigenvalues λ_s.

The following explicit relation applies to the i-th component of the eigenvector which belongs to the eigenvalue λ_s

$$x_{is} = l_s b_1^{-1} b_2^{-1} \cdots b_{i-1}^{-1} D_{i-1}(\lambda_s) \quad (i,\ s = 1,\ 2,\ \dots,\ n). \tag{10}$$

Let us now consider the matrix P'. It is quite evident that its characteristic polynomial represents an expansion based also on the recurrence formulas (9), while its eigenvectors are calculated by the formula

$$y_{is} = \overline{l}_i c_2^{-1} c_3^{-1} \cdots c_s^{-1} D_{s-1}(\lambda_i) \quad (s,\ i = 1,\ 2,\ \dots,\ n). \tag{11}$$

The orthogonality relation for the eigenvectors of matrices P and P' holds true

$$x_{1s}y_{i1} + x_{2s}y_{i2} + \cdots + x_{ns}y_{in} = \begin{cases} 0, & \text{if } i \neq s, \\ l_s\overline{l}_i d_i, & \text{if } i = s. \end{cases}$$

Assuming $l_s = 1$ and $\overline{l}_i = 1/d_i$, we obtain orthonormalized systems of vectors X_s and Y_i.

The matrix constructed with eigenvectors (row vectors) Y_i of matrix P' will be the sought matrix G.

The problem (6) stated in the canonical form (8) is

$$U_i' + \lambda_i U_i = F_i(t),\ U_i(0) = \Psi_i \quad (i = 1,\ 2,\ \ldots,\ n).$$

In this way we will have n independent Cauchy problems for first-order equations. The solutions to these problems are

$$U_i(t) = \Psi_i \exp(-\lambda_i t) + \int_0^t \exp[-\lambda_i(t-\tau)] F_i(\tau) d\tau \quad (i = 1,\ 2,\ \ldots,\ n),$$

where

$$F_i(t) = \sum_{s=1}^{n} d_i^{-1}[f_s(t) c_2^{-1} \cdots c_s^{-1} D_{s-1}(\lambda_i)] - d_i^{-1}[c_1\varphi_1(t) + b_n\varphi_2(t) c_2^{-1} \cdots c_n^{-1} D_{n-1}(\lambda_i)],$$

$$\Psi_i = \sum_{s=1}^{n} d_i^{-1}\psi_i c_2^{-1} \cdots c_s^{-1} D_{s-1}(\lambda_i).$$

Performing the inverse G-transformation will yield the solution of problem (4), (5):

$$u_i(t) = \sum_{k=1}^{n}\left[\Psi_k \exp(-\lambda_k t) + \int_0^t \exp[-\lambda_k(t-\tau)] F_k(t) d\tau\right] b_1^{-1} \cdots b_{i-1}^{-1} D_{i-1}(\lambda_k). \tag{12}$$

Using the explicit formulas (9), (10) for determining the eigenvectors of matrices P and P' is not expedient when the value of n is large [5]. The eigenvector which corresponds to the eigenvalue λ_s can be found easily by solving the respective system of equations. The appearance of matrices P and P' indicates that the first and the last equation of this system contain two unknowns while all the other equations contain three each. Inasmuch as the eigenvector is determined with an accuracy up to an arbitrary factor, with the arbitrary first component x_{1s} given it becomes possible to determine all the others by a subsequent solution of one equation with one unknown.

As expression (12) shows, the solution obtained for problem (1), (2), (3) contains an analytic statement with respect to the variable t, and here this method offers definite advantages over other numerical methods.

In order to establish whether problem (4), (5) is amenable to an approximation of the exact solution, one must find out whether the system of differential-difference equations has a unique and stable solution.

It follows from (12) that a solution to the Cauchy problem (4), (5) exists, if functions f(x, t), $\varphi_1(t)$, and $\varphi_2(t)$ are continuous in the interval [0, T].

We will now establish the uniqueness of the differential-difference equations system. For this purpose we designate

$$(\alpha,\ \beta) = \sum_{i=1}^{n} \alpha_i\beta_i h_i,\ (\alpha,\ \beta] = \sum_{i=1}^{n+1} \alpha_i\beta_i h_i,\ (\alpha,\ \beta)^* = \sum_{i=1}^{n} \alpha_i\beta_i \hbar_i,$$

$$z_{\bar{x},i} = \frac{z_i - z_{i-1}}{h_i},\ z_{x,i} = \frac{z_{i+1} - z_i}{h_{i+1}},\ a_{i+1} = k_{i+1/2},\ a_i = k_{i-1/2},$$

$$(az_{\bar{x}})_{\tilde{x},i} = \frac{1}{\hbar_i}\left[a_{i+1}\frac{z_{i+1} - z_i}{h_{i+1}} - a_i\frac{z_i - z_{i-1}}{h_i}\right].$$

In these designations there is implied Green's difference formula for a nonuniform grid [6]:

$$(y,\ (az_{\bar{x}}))_{\tilde{x}}^* = -(a,\ y_{\bar{x}} z_{\bar{x}}] + (ayz_{\bar{x}})_{n+1} - a_1(y_x z)_0. \tag{13}$$

Assuming the existence of two solutions u_{1i}, u_{2i} (i = 1, 2, . . . , m) and considering the difference $v_i = u_{1i} - u_{2i}$ (i = 1, 2, . . . , n) which satisfies a system of homogeneous equations with homogeneous initial conditions, we will prove that functions $v_i(t)$ are identically equal to zero.

Consider the functional

$$J(t) = 2\sum_{i=1}^{n}\int_0^t \rho_i v_i'^2 \hbar_i d\tau + \sum_{i=1}^{n} k_{i+1/2}\left(\frac{v_i - v_{i-1}}{h_i}\right)^2 h_i + \sum_{i=1}^{n} q_i v_i^2 \hbar_i. \tag{14}$$

Differentiating (14) with respect to t, applying formula (13) and homogeneous conditions, we obtain

$$\frac{dJ(t)}{dt} = 2\sum_{i=1}^{n}\left\{\rho_i v_i' - \frac{1}{\hbar_i}\left[k_{i+1/2}\frac{v_{i+1}-v_i}{h_{i+1}}\right.\right.$$

$$\left.\left. - k_{i-1/2}\frac{v_i - v_{i-1}}{h_i}\right] + q_i v_i\right\}\hbar_i v_i' + 2k_{n+1/2}v_{n+1}'\frac{v_{n+1}-v_n}{h_{n+1}} - 2k_{1/2}v_0'\frac{v_1 - v_0}{h_1} = 0.$$

If follows from here that $J(t) = c_1 \equiv \text{const}$ and, since $J(0) = 0$, that $J(t) = 0$. This is equivalent to the conditions:

$$v_i'(t) = 0, \quad v_i - v_{i-1} = 0 \quad (i = 1, 2, \ldots, n). \tag{15}$$

Taking into account (15) and homogeneous initial conditions, it is easy to show that $v_i(t) = \text{const} \equiv 0$ $(i = 1, 2, \ldots, n)$.

Let

$$\rho(x) \geqslant m_\rho > 0, \quad k(x) \geqslant m_k > 0, \quad q(x) \geqslant 0, \quad h_0 = \max_{1 \leqslant i \leqslant n} h_i,$$

$$k_0 = \max_{x\in[0,\,b]}|k(x)|, \quad k_1 = \max_{x\in[0,\,b]}|k'(x)|, \quad k_2 = \max_{x\in[0,\,b]}|k''(x)|,$$

$$A_1 = \max_{x\in[0,\,b],\,t\in[0,\,T]}|u_x'(t,\,x)|, \quad A_2 = \max_{x\in[0,\,b],\,t\in[0,\,T]}|u_{xx}''(x,\,t)|, \quad A_3 = \max_{x\in[0,\,b],\,t\in[0,\,T]}|u_{xxx}'''(x,\,t)|.$$

The error of the solution can then be estimated, as in [7-9], to be

$$|\gamma_i(t)| \leqslant h_0 M\sqrt{2\frac{(b-x_i)x_i t}{m_\rho m_k}} \qquad (i = 1, 2, \ldots, n; \quad 0 \leqslant t \leqslant T),$$

where

$$\gamma_i(t) = u(x_i,\,t) - u_i(t), \quad M = k_2 A_1 + k_1 A_2 + \frac{2}{3}k_0 A_3.$$

This estimate determines the convergence of the straight-lines method.

The results shown here can be extended to the heat conduction problem with other linear boundary conditions, and also to the case of discontinuous coefficients in the equation.

Particularly, the straight-lines method outlined here with a nonuniform step is convenient to use for problems relating to the propagation of heat through multilayer media.

NOTATION

u	is the temperature;
$\rho(x)$	is the volume heat capacity;
$k(x)$	is the heat conductivity;
$q(x)$	is the heat transfer coefficient;
$f(x, t)$	is the density of heat sources at a point x at an instant of time t;
$\varphi_1(t)$, $\varphi_2(t)$	are the boundary-value functions;
'	is the sign of transposition.

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